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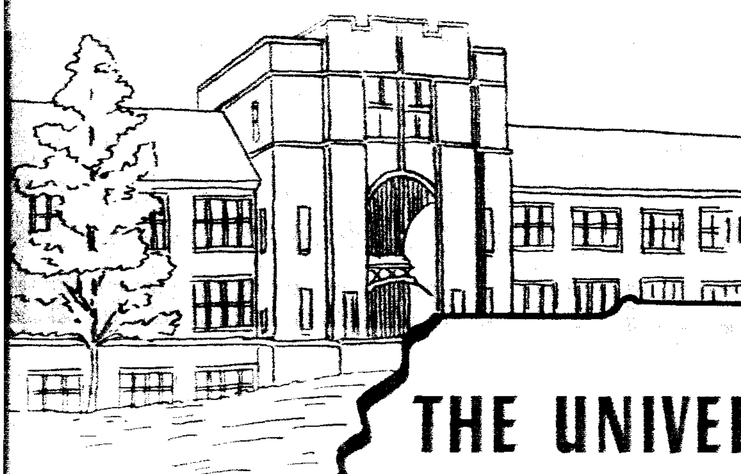

THE UNIVERSITY OF TENNESSEE

DEPARTMENT OF ELECTRICAL ENGINEERING

DESIGN OF MINIMUM ENERGY DISCRETE-DATA CONTROL SYSTEMS

by

A. M. Revington and J. C. Hung

Supported by National Aeronautics and Space Administration
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SUMMARY

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A procedure is developed for designing the minimum energy discrete-data control of an n-th order continuous plant. The system is required to reach a desired final state from a given initial state in N sampling periods, with $N > n$. A very useful matrix, called derived matrix, is developed, which expresses the relationship between the canonical vectors. The minimum energy controller depends only on the derived matrix, a very simple result. The importance of minimum energy control is discussed. An example is given to demonstrate the method.

Author

INTRODUCTION

The availability of modern high speed digital computers as controllers in a feedback control system demands a new approach to control system design involving discrete-data. In recent years considerable effort has been expended on the optimum design of discrete-data control systems, especially on time-optimal deadbeat controls.

For systems without saturation, Kalman and Bertram presented a very elegant method[1]. Kalman also proposed a method for saturating time-optimal control[2].

It is well known[1] that for non-saturating time-optimal control an n-th order system can be brought from an initial state to a desired final state in n sampling periods or less. But for saturating control the required number of sampling periods is, in general, greater than n. Furthermore, the control is not unique. A unique control can be obtained by imposing an additional constraint, for example, the so-called minimum energy control.

In many practical problems, the system output is required to be error-free after a finite period $NT > nT$, but the time-optimal response is not necessary. In this paper a procedure is developed for designing the minimum sum-of-input-squares, conventionally called minimum energy, discrete-data control of an n-th order continuous plant preceded by a zero-order hold. That is we want

$$E = \sum_{k=1}^{N} \left| m(k) \right|^2 = \text{minimum} \tag{1}$$

where m(k) is the system input during the period $(\overline{k-1}T, kT)$. The system is required to reach the origin of the state space from a given initial state in N sampling periods, with $N > n$. A important matrix, called the derived matrix, is developed, which expresses the relationship between the canonical vectors. The minimum energy control depends on the derived matrix only, which is in a very simple form and is easy to implement. An example is given to illustrate the method. The method is not restricted by the order of the system. The importance of this type of control is discussed.

SYMBOLS

$\underline{a}$	n-vector whose components are $a_1, a_2, \ldots., a_n$
a_i, $i = 1, 2, \ldots.$	real constants
$\underline{b}$	(N-n)-vector whose components are $a_{n+1}, \ldots, a_N$
$\underline{c}$	n-vector representing the system state in the canonical space C
c_i, $i = 1, 2, \ldots, n$	real constants representing the components of $\underline{c}$
C	n-dimensional canonical space of the system
E	sum of the squares of the system input steps
G	system transistion matrix
$\underline{h}$	n-dimensional forcing vector of the system
H	derived matrix
H^t	Transpose of H
h_{ij}	ij-th element of H
i, j, k	running indexes
m(k)	system input during the period $(\overline{k-1}T, kT)$

$\underline{m}$	N-vector whose components are m(1), m(2), ..., m(N)
n	order of the system to be controlled
N	number of sampling periods at the end of which the system is required to reach its desired final state
Q	nx(N-n) matrix whose columns are the canonical vectors $\underline{r}_{n+1}$, $\underline{r}_{n+2}$, ..., $\underline{r}_N$.
$\underline{r}_i$, i = 1, 2, ..., N	canonical vectors of the system
R	nxn matrix whose columns are the canonical vectors $\underline{r}_1$, $\underline{r}_2$, ..., $\underline{r}_n$
R^{-1}	inverse of R
$\underline{s}_i$, i = 1, 2, ..., n	unit vectors of the canonical space
t	continuous time variable
T	Sampling period
$\underline{x}$	n-vector representing the state of the system in the state space
X	n-dimensional state space of the system

MATHEMATICAL FORMULATION

Consider an n-th order, linear, time-invariant, sampled-data system. Let $\underline{x}(k)$ be an n-vector representing the state of the system at time kT, where T is sampling period. The system state transition equation is

$$\underline{x}(k+1) = g(T)\underline{x}(k) + \underline{h}(T)m(k+1) \tag{2}$$

where G(T) is the nxn system transition matrix, $\underline{h}(T)$ is the n-dimensional forcing vector, and m(k+1) is the constant control during period (kT, $\overline{k+1}$T). G(T) has the following properties.

$$G(t_1+t_2) = G(t_1)G(t_2) \tag{3}$$

$$G^{-1}(T) = G(-T) \tag{4}$$

Define the canonical vectors of the system as

$$\underline{r}_i = G^{-i}(T)\underline{h}(T) = G(-iT)\underline{h}(T) \ , \quad i \ , 1, 2, \ldots. \tag{5}$$

For a controllable system[5], we can choose the first n $\underline{r}_i$'s to form a basis for the n-dimensional state space X, i.e., any state $\underline{x}$ in X can be expressed by

$$\underline{x} = \sum_{i=1}^{n} a_i \underline{r}_i \tag{6}$$

where a_i's are real constants.

The control sequence for time-optimal control[2] without saturation is given by

$$m(k) = -a_k \qquad k = 1, 2, \ldots\ldots, n \tag{7}$$

This control is unique. As a consequence, the input energy, Eq. (1), is unique for a given initial state.

Very often, time optimal control is unnecessary, rather the system is required to reach the desired final state in finite time, say, N sampling periods with $N > n$. Under this situation each state $\underline{x}$ in X can be expressed as the linear combination of N canonical vectors, i.e.,

$$x = \sum_{i=1}^{N} a_i \underline{r}_i \tag{8}$$

and the control sequence is given by

$$m(k) = -a_k \qquad k = 1, 2, \ldots\ldots, N \tag{9}$$

Note that in Eq. (8) only n of the N $\underline{r}_i$'s are linearly independent. Hence, by choosing the a_i's differently, there are an infinite number of different linear combinations possible. As a consequence, there are an infinite number of possible controls, Eq. (9), which will bring the system from the initial state to the final state in N sampling periods.

We want to single out, from the infinite number of possible controls, the one which consumes minimum energy, i.e.,

$$E = \sum_{k=1}^{N} |m(k)|^2 = \sum_{k=1}^{N} a_i^2 = \text{minimum} \tag{10}$$

This case has great practical importance due to the following facts. First, the constraint imposed by Eq. (10) restrains the larger control magnitudes and indulges the smaller ones, thus the system is less likely to saturate. Secondly, the same mission is accomplished but with the least energy. Thirdly, the procedure of obtaining the optimum control for minimum energy is much simpler than those for saturated control and/or so-called minimum fuel control[6].

THE DERIVED MATRIX AND THE CANONICAL SPACE

We have seen in the last section that, for $N > n$, only n of the N canonical vectors are independent. If we choose the first n vectors, $\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n$, as a basis the remaining N-n vectors can be written as

$$\underline{r}_{n+j} = \sum_{i=1}^{n} h_{ij}\, \underline{r}_i \qquad j = 1, 2, \ldots, N-n \tag{11}$$

In matrix form

$$\left[\underline{r}_{n+1}, \underline{r}_{n+2}, \ldots \underline{r}_N\right] = \left[\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n\right] \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1(N-n)} \\ h_{21} & h_{22} & \cdots & h_{2(N-n)} \\ \vdots & & & \\ h_{n1} & h_{n2} & \cdots & h_{n(N-n)} \end{bmatrix} \tag{12}$$

The nx(N-n) matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1(N-n)} \\ h_{21} & h_{22} & \cdots & h_{2(N-n)} \\ \vdots & & & \\ h_{n1} & h_{n2} & \cdots & h_{n(N-n)} \end{bmatrix} \tag{13}$$

is named "derived matrix," which indicates the linear dependence relationship between vectors $\underline{r}_1$, ... $\underline{r}_n$ and vectors $\underline{r}_{n+1}$, ... $\underline{r}_N$.

Let R be an nxn matrix whose columns are vectors $\underline{r}_1$, $\underline{r}_2$, ... $\underline{r}_n$ and let Q be an nx(N-n) matrix whose columns are vectors $\underline{r}_{n+1}$, $\underline{r}_{n+2}$, ... $\underline{r}_n$. Then Eq. (12) becomes

$$Q = R\ H \tag{14}$$

Further, let us define two vectors

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{15}$$

$$\underline{b} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_N \end{bmatrix} \tag{16}$$

Using these notations, any state $\underline{x}$ in X may be put into the form

$$\begin{aligned} \underline{x} &= \sum_{i=1}^{n} a_i\ \underline{r}_i + \sum_{j=1}^{N-n} a_{n+j}\ \underline{r}_{n+j} \\ &= R\ \underline{a} + Q\ \underline{b} \\ &= R\ \underline{a} + R\ H\ \underline{b} \end{aligned} \tag{17}$$

For future convenience, we form a new n-dimensional space C, called "canonical space," whose unit vectors, $\underline{s}_1$, $\underline{s}_2$, ..., $\underline{s}_n$, correspond to the canonical vectors, $\underline{r}_1$, $\underline{r}_2$, ..., $\underline{r}_n$, in the state space X. This amounts to applying a linear transformation R^{-1} to Eq. (17), giving

$$\underline{c} = \underline{a} + H\,\underline{b} \tag{18}$$

where $\underline{c} = R^{-1}\underline{x}$ is an n-vector in C corresponding to the state $\underline{x}$ in X. Each component of $\underline{c}$ gives

$$c_i = a_i + \sum_{j=1}^{N-n} a_{n+j}\, h_{ij} \qquad i = 1, 2, \ldots., n \tag{19}$$

Eq. (18) expresses the system initial state in the canonical space in terms of the control sequence. This equation is very useful for the future development.

MINIMUM ENERGY RELATIONSHIP

For a given initial state $\underline{c}$ in C (or, equivalently, $\underline{x}$ in X) and for a given control period NT with $N > n$, the control sequence is completely fixed once N-n of the N control steps are chosen. Now, we want to choose $m(k) = -a_k$ in such a way that Eq. (10) is satisfied.

From Eq. (19)

$$a_i = c_i - \sum_{j=1}^{N-n} h_{ij}\, a_{n+j} \qquad i = 1, \ldots., n \tag{20}$$

The total control energy is

$$E = \sum_{i=1}^{N} |m(i)|^2 = \sum_{i=1}^{N} a_i^2$$

$$= \sum_{i=1}^{n} \left[c_i - \sum_{j=1}^{N-n} h_{ij}\, a_{n+j} \right]^2 + \sum_{j=1}^{N-n} a_{n+j}^2 \tag{21}$$

Minimizing Eq. (21) with respect to a_{n+1}, ..., a_N, gives the following important relationship. (Appendix I)

$$a_{n+j} = \sum_{i=1}^{n} h_{ij} a_i \qquad j = 1, 2, \ldots, N-n \tag{22}$$

The matrix form of Eq. (22) is

$$\underline{b} = H^t \, \underline{a} \tag{23}$$

where H^t is the transpose of H.

Combining Eqs. (18) and (23),

$$\underline{c} = \left[I + H\,H^t \right] \underline{a} \; . \tag{24}$$

The optimum choice of a is therefore

$$\underline{a} = \left[I + H\,H^t \right]^{-1} \underline{c} \tag{25}$$

Eqs. (25) and (23) give the solution for the minimum energy control, which show that the controller depends on the derived matrix H only. By substituting these two equations into Eq. (21) the minimum energy is (Appendix II)

$$E = \underline{c}^t \, \underline{a} = \underline{x}^t \left[R^{-1} \right]^t \underline{a} \; . \tag{26}$$

DESIGN PROCEDURE

The above results are summarized in the following as the design procedure.

1. Given a linear, time-invariant, dynamical system and the sampling period T, the transition matrix G(t) and forcing vector $\underline{h}(t)$ are obtained.

2. Compute the canonical vectors

$$\underline{r}_i = G(-iT)\underline{h}(T) \qquad i = 1, 2, \ldots, N \tag{27}$$

3. Write the matrices

$$R = \left[\underline{r}_1, \underline{r}_2, \ldots\ldots, \underline{r}_n\right] \tag{28}$$

and

$$Q = \left[\underline{r}_{n+1}, \ldots\ldots, \underline{r}_N\right] \tag{29}$$

4. Compute the derived matrix

$$H = R^{-1} Q \tag{30}$$

5. Compute the control vector

$$\underline{m} = \begin{bmatrix} m(1) \\ m(2) \\ \vdots \\ \vdots \\ m(N) \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \\ -a_{n+1} \\ \vdots \\ -a_N \end{bmatrix} = \begin{bmatrix} \underline{a} \\ \\ \underline{b} \end{bmatrix} \tag{31}$$

where

$$\underline{a} = \left[I + H\,H^t\right]^{-1} \underline{c} = \left[I + H\,H^t\right]^{-1} R^{-1} \underline{x} \tag{32}$$

$$\underline{b} = H^t\,\underline{a} \tag{33}$$

Fig. 1 shows the block diagram of the complete control system.

EXAMPLE

Consider the discrete control of a second order system having a transfer function

$$F(s) = \frac{Y(s)}{M(s)} = \frac{1}{s(s+1)} \tag{34}$$

The sampling period is T = 1 second. It is desired to bring the system from an initial state to the origin in N = 4 sampling periods.

Choosing state variables $x_1 = y$ and $x_2 = \dot{y}$ we have the vector differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m \tag{35}$$

1. The solution of Eq. (35) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} e^{-t}+t-1 \\ 1-e^{-t} \end{bmatrix} m(t) \tag{36}$$

Thus, the transition matrix and forcing vector are, respectively,

$$G(t) = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix} \tag{37}$$

$$\underline{h}(t) = \begin{bmatrix} e^{-t} + t - 1 \\ 1 - e^{-t} \end{bmatrix} \tag{38}$$

2. The canonical vectors are

$$\underline{r}_i = G(-iT)\underline{h}(T) = G(-i)\underline{h}(1) = \begin{bmatrix} 1 & 1-e^{i} \\ 0 & e^{i} \end{bmatrix} \begin{bmatrix} e^{-1} \\ 1-e^{-1} \end{bmatrix}$$

or

$$\underline{r}_i = \begin{bmatrix} 1 + e^{i-1} - e^{i} \\ e^{i} - e^{i-1} \end{bmatrix} \qquad i = 1, 2, \ldots. \tag{39}$$

3.
$$R = \left[\underline{r}_1, \ \underline{r}_2\right] = \begin{bmatrix} -0.7182 & -3.6706 \\ 1.7182 & 4.6706 \end{bmatrix} \tag{40}$$

$$Q = \left[\underline{r}_3, \ \underline{r}_4\right] = \begin{bmatrix} -11.6961 & -33.5118 \\ 12.6961 & 34.5118 \end{bmatrix} \tag{41}$$

4.
$$H = R^{-1}Q = \begin{bmatrix} -2.7183 & -10.1074 \\ 3.7183 & 11.1074 \end{bmatrix} \tag{42}$$

5.
$$\left[I + H\ H^{t}\right]^{-1} = \begin{bmatrix} 0.457091 & 0.404748 \\ 0.404748 & 0.365635 \end{bmatrix} \tag{43}$$

$$\left[I + H\ H^{t}\right]^{-1} R^{-1} = \begin{bmatrix} 0.48756 & 0.46983 \\ 0.42751 & 0.41427 \end{bmatrix} \tag{44}$$

$$\underline{a} = \begin{bmatrix} 0.48756 & 0.46982 \\ 0.42751 & 0.41427 \end{bmatrix} \underline{x} \tag{45}$$

$$\underline{b} = \begin{bmatrix} -2.7183 & 3.7183 \\ -10.1074 & 11.1074 \end{bmatrix} \underline{a} \tag{46}$$

For any given initial state in state space X, Eqs. (45) and (46) give the control sequence which brings the system state to the origin in 4 sampling periods. For example, if the initial state is $x_1(0) = 1$ and $x_2(0) = 0$ (corresponds to a unit step input), then the control sequence is given by

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left[I + H\ H^{t}\right]^{-1} R^{-1}\underline{x} = \begin{bmatrix} 0.48756 \\ 0.42756 \end{bmatrix} \tag{47a}$$

$$\underline{b} = \begin{bmatrix} a_3 \\ \\ a_4 \end{bmatrix} = H^t \underline{a} = \begin{bmatrix} 0.2643 \\ \\ -0.1795 \end{bmatrix} \qquad (47b)$$

It is interesting to compare the result to that of time-optimal non-saturating control whose control sequence is given by

$$\underline{a} = \begin{bmatrix} a_1 \\ \\ a_2 \end{bmatrix} = R^{-1} \underline{x} = \begin{bmatrix} 1.5820 \\ \\ -0.5820 \end{bmatrix} \qquad (48)$$

However, if saturation does occur, say, at m = 1, then the actual control sequence for the time-optimal design becomes

$$a = \begin{bmatrix} a_1 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} 1.000 \\ \\ -0.5820 \end{bmatrix} \qquad (49)$$

and the system output becomes retarded, whereas the control sequence for minimum energy control is not affected, at least for this example. Fig. 2 shows the three output response curves of the system for (a) minimum energy design with N = 4, Eqs. 47a and 47b; (b) ideal time-optimal design which has N = 2, Eq. (48); and (c) retarded time-optimal design, Eq. (49). We immediately see that although minimum energy design takes a longer time to settle, it does give an error-free response. Fig. 3 gives the system inputs for the three cases.

SOME REMARKS

A note on the choice of the number of sampling periods N would be in order. One might think that a longer N could result in a smaller minimum energy. This is indeed the case since the set of all input sequences of length N+k contains input sequences of length N when the last k inputs are set to zero. Thus any minimum energy sequence longer than N consumes, at the most, only the minimum energy required for the sequence of length N.

As pointed out earlier and illustrated by the example, the minimum energy design also tends to prevent the system from saturation. This method is especially effective when N is large. In many practical systems, where the time-optimal control is not essential, it is desirable to choose

N large to avoid saturation. As a matter of fact, if N is not large enough to prevent saturation in this minimum energy design then most likely any other design technique with the same N will also be unable to prevent saturation.

CONCLUSIONS

A procedure has been developed for designing the minimum energy error-free discrete-data control for an n-th order continuous plant. The system is required to reach a desired final state from a given initial state in N sampling periods, with $N > n$. An important matrix, called derived matrix, has been developed, which expresses the relationship between the canonical vectors. The minimum energy controller has been shown to depend only on the derived matrix. The design procedure is simple and easy to implement. The importance of minimum energy control and the choice of the number of sampling period, N, have been discussed. An example has been given to illustrate the method.

An extension of the theory to time-weighted minimum energy control will be given in a forthcoming paper. Further research is underway on the relations between the number of sampling periods, the sampling period, the initial condition and the minimum energy.

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APPENDICES

I. Derivation of Minimum Energy Relationship Eq. (23)

Eq. (21) is

$$E = \sum_{i=1}^{n} \left[c_i - \sum_{j=1}^{N-n} h_{ij}\, a_{n+j} \right]^2 + \sum_{j=1}^{N-n} a_{n+j}^2 \qquad \text{(A-1)}$$

Differentiating this equation with respect to a_{n+p}, with $p = 1, 2, \ldots,$ N-n,

$$\frac{\partial E}{\partial a_{n+p}} = -2 \sum_{i=1}^{n} h_{ip} \left[c_i - \sum_{j=1}^{N-n} h_{ij}\, a_{n+j} \right] + 2\, a_{n+p} \qquad \text{(A-2)}$$

Equating Eq. (A-2) to zero and rearranging the terms,

$$\sum_{i=1}^{n} h_{ip}\, c_i = a_{n+p} + \sum_{j=1}^{N-n} a_{n+j} \sum_{i=1}^{n} h_{ip}\, h_{ij} \qquad \text{(A-3)}$$

The above equation can be expressed in its matrix form with the aid of Eqs. (13) and (16).

$$H^t\, \underline{c} = \underline{b} + H^t\, H\, \underline{b} \qquad \text{(A-4)}$$

Multiplying H^t to Eq. (18), we also have

$$H^t\, \underline{c} = H^t\, \underline{a} + H^t\, H\, \underline{b} \qquad \text{(A-5)}$$

Comparing Eqs. (A-4) and (A-5) gives

$$\underline{b} = H^t\, \underline{a} \qquad \text{(A-6)}$$

which is Eq. (23)

II. Derivation of Minimum Energy Eq. (26)

The total input energy is

$$E = \sum_{i=1}^{N} a_i^2 = \underline{a}^t \, \underline{a} + \underline{b}^t \, \underline{b} \tag{A-7}$$

Using Eq. (A-6) and (18) gives

$$\begin{aligned} E &= \underline{a}^t \, \underline{a} + \underline{b}^t \, H^t \, \underline{a} \\ &= \left[\underline{a}^t + \underline{b}^t \, H^t \right] \underline{a} \\ &= \underline{c}^t \, \underline{a} \end{aligned} \tag{A-8}$$

which is Eq. (26)

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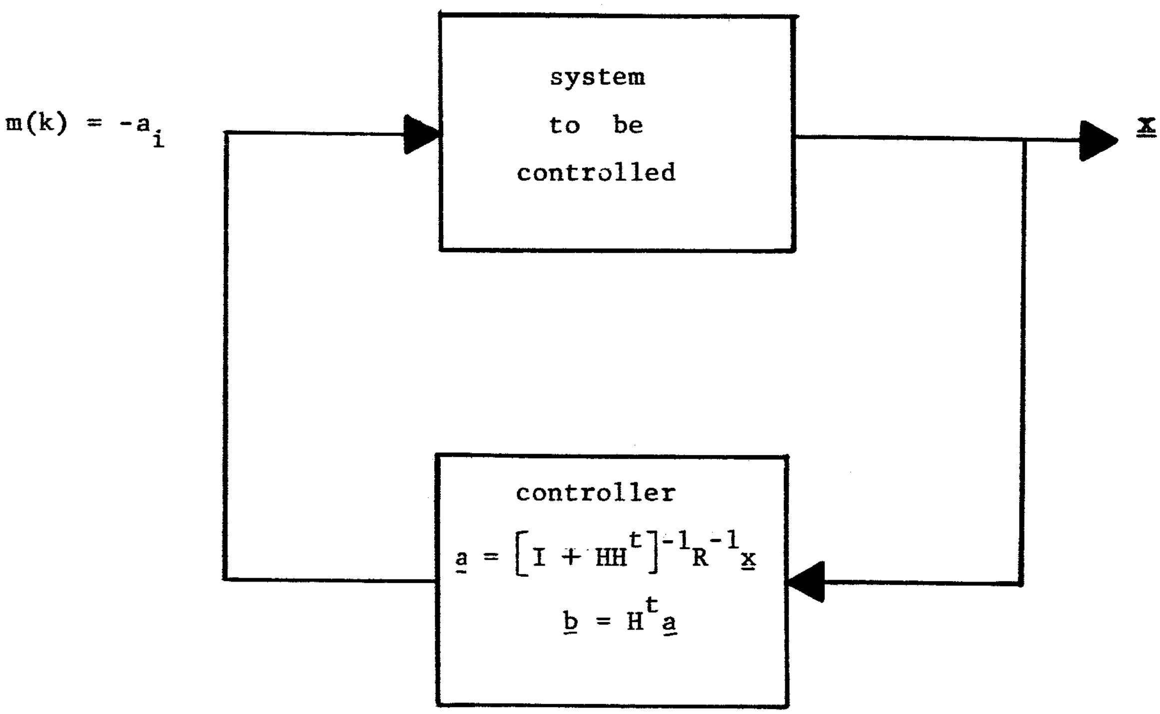

Fig. 1 Block diagram of control system

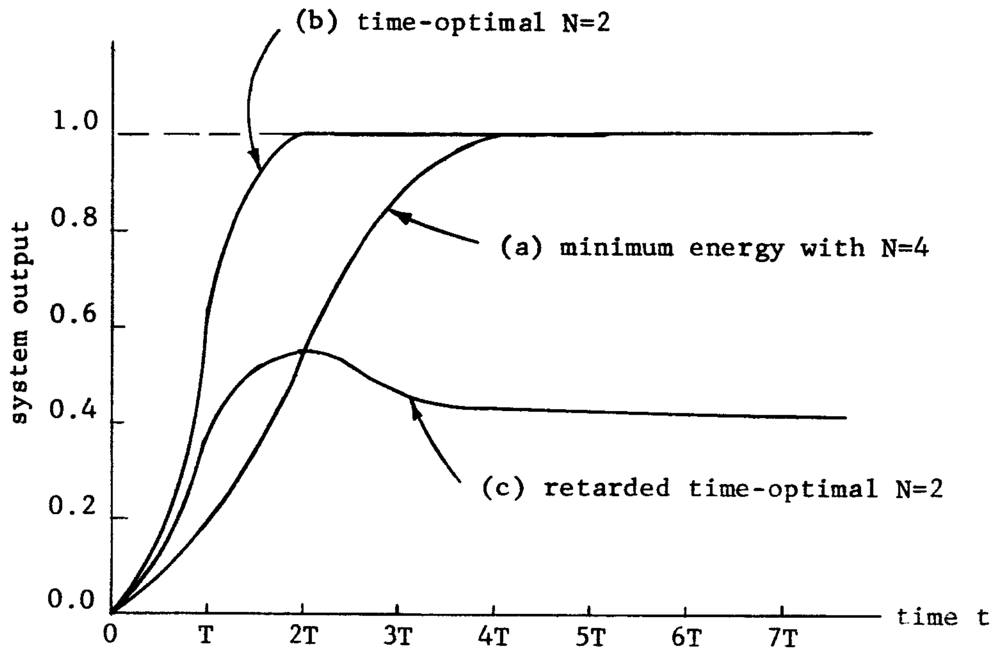

Fig. 2 System output responses

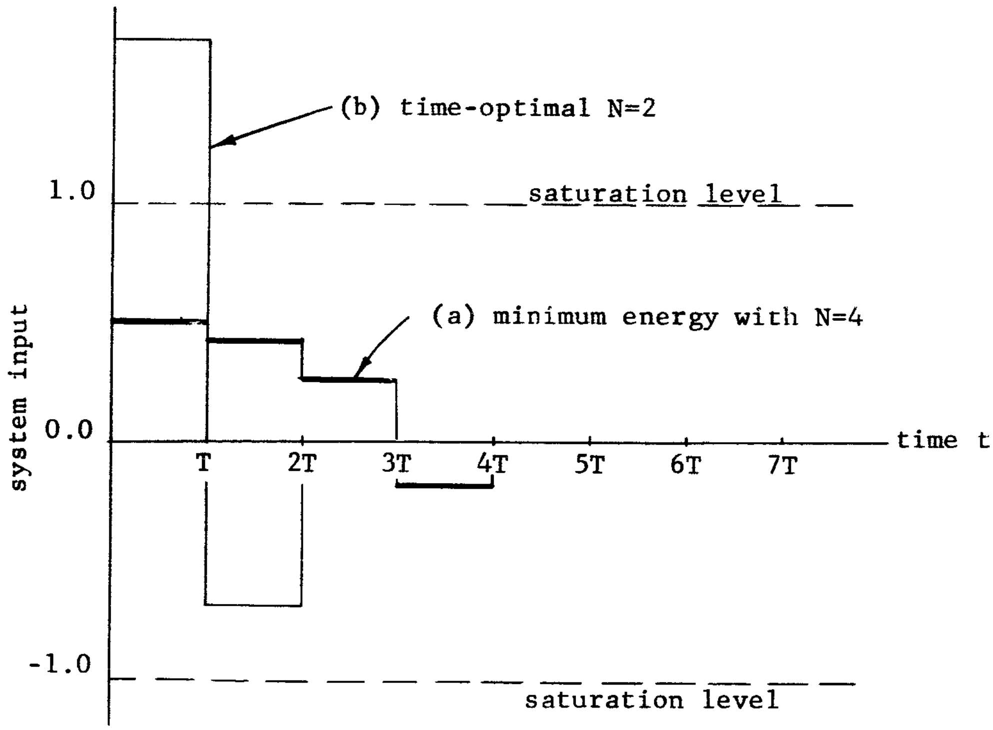

Fig. 3 System inputs